The minimal size of a graph with generalized connectivity $\kappa_3 = 2$*

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Abstract

Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. Chartrand et al. generalized the concept of connectivity as follows: The *k-connectivity*, denoted by $\kappa_k(G)$, of G is defined by $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G.

This paper mainly focuses on the minimal number of edges of a graph G with $\kappa_3(G) = 2$. For a graph G of order $v(G)$ and size $e(G)$ with $\kappa_3(G) = 2$, we obtain that $e(G) \geq \frac{6}{5}v(G)$, and the lower bound is sharp by showing a class of examples attaining the lower bound.

Keywords: k-connectivity; internally disjoint trees

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1 Introduction

We follow the terminology and notation of [1] and all graphs considered here are always simple. As usual, we denote the numbers of vertices and edges in G by $v(G)$ and $e(G)$,

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and these two basic parameters are called the *order* and *size* of G, respectively. Let X be a set of vertices of G and $G[X]$ the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. A stable set in a graph is a set of vertices no two of which are adjacent. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [4] provides an equivalent definition of the connectivity. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

In [2], the authors generalized the concept of connectivity. Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \backslash S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called an *internally disjoint set of trees connecting* S. The *k-connectivity*, denoted by $\kappa_k(G)$, of G is then defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus, $\kappa_2(G) = \kappa(G)$.

In [3], we focused on the investigation of $\kappa_3(G)$ and mainly studied the relationship between the 2-connectivity and the 3-connectivity of a graph. We gave sharp upper and lower bounds for $\kappa_3(G)$ for general graphs G, and showed that if G is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$. Moreover, we studied the algorithmic aspects for $\kappa_3(G)$ and gave an algorithm to determine $\kappa_3(G)$ for a general graph G.

In this paper, we will turn to determining the minimal number of edges of a graph G with $\kappa_3 = 2$. For a graph G of order $v(G)$ and size $e(G)$ with $\kappa_3(G) = 2$, we obtain that $e(G) \geq \frac{6}{5}v(G)$, and the lower bound is sharp by constructing a class of graphs which attain the lower bound. Note that for a graph G of order $v(G)$ and size $e(G)$ with $\kappa(G) = 2$, we only have $e(G) \geq v(G)$, and a cycle of this order attains the lower bound.

2 Lower bound

Before proceeding, we recall a result in [3], which will be used frequently in the sequel.

Lemma 2.1. *If G is a connected graph with minimum degree δ, then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Now we give the lower bound.

Proposition 2.1. *Every graph G of order n with $\kappa_3(G) = 2$ has at least $\frac{6}{5}n$ edges.*

Proof. Since $\kappa_3(G) = 2$, by Lemma 2.1, we know that $\delta(G) \geq 2$ and any two vertices of degree 2 are not adjacent. Denote by X the set of vertices of degree 2. By Lemma 2.1, we have that X is a stable set. Put $Y = V(G) - X$ and obviously there are $2|X|$ edges joining X to Y. Assume that m' is the number of edges joining two vertices belonging to Y. It is clear that

$$e = 2|X| + m'. \tag{1}$$

Since every vertex of Y has degree at least 3 in G, then $\sum_{v \in Y} d(v) = 2|X| + 2m' \geq 3|Y| = 3(n - |X|)$, namely,

$$5|X| + 2m' \geq 3n. \tag{2}$$

Combining (1) with (2), we have $\frac{5}{2}e = \frac{5}{2}(2|X| + m') = 5|X| + \frac{5}{2}m' \geq 5|X| + 2m' \geq 3n$, namely, $e \geq \frac{6}{5}n$. The proof is complete. ∎

Remark 2.1: Furthermore, in Proposition 2.1 equality holds if and only if $5|X| + \frac{5}{2}m' = 5|X| + 2m' = 3n$, namely, if and only if

(A) $m' = 0$, that is, Y is a stable set and

(B) the maximum degree Δ is 3.

Moreover, when equality holds, inequality (2) becomes $5|X| = 3n$, that is, $|X| = \frac{3}{5}n$.

Remark 2.2: Obviously, for any graph G with $e(G) = \frac{6}{5}v(G)$, $\kappa_3(G) \leq 2$. The next lemma shows that the number $e(G) = \frac{6}{5}v(G)$ cannot guarantee that $\kappa_3(G) = 2$.

Lemma 2.2. *For any connected graph G of order 10 and size 12, $\kappa_3(G) = 1$.*

Proof. Note that $e(G) = \frac{6}{5}v(G)$ and so $\kappa_3(G) \leq 2$. Assume, to the contrary, that there is a connected graph G of order 10 and size 12 with $\kappa_3(G) = 2$. Therefore by Remark 2.1, both X and Y are stable sets, $|X| = \frac{3}{5}v(G) = 6$ and $|Y| = 4$, where X and Y are the sets of vertices of degrees 2 and 3, respectively. Let $X = \{x_1, \ldots, x_6\}$ and $Y = \{y_1, \ldots, y_4\}$.

Case 1: For every two vertices y_i and y_j in Y, there is a vertex in X that is adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 6$.

Note that every vertex in X has degree 2 and there are exactly six 2-subsets of Y, namely

$$\{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_4\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_3, y_4\}.$$

Thus we may assume that G is isomorphic to Figure 1. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_2 and x_4, contrary to our assumption.

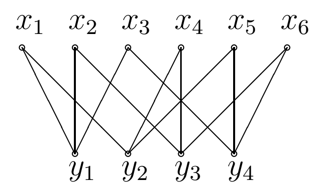

Figure 1: The graph for Case 1 of Lemma 2.2

Case 2: For some two vertices y_i and y_j in Y, at least two vertices in X are adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 6$. Since G is connected, we can get that only two vertices in X are adjacent to both y_i and y_j. Then we may assume that G is isomorphic to Figure 2. Now consider the three vertices x_1, x_3 and x_5 and we can get $\kappa_3(G) = 1$, contrary to our assumption.

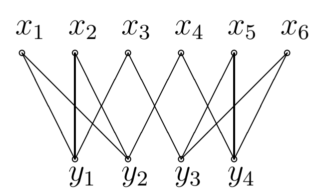

Figure 2: The graph for Case 2 of Lemma 2.2

The proof is complete. ∎

Next we will show that the lower bound given in Proposition 2.1 is essentially best possible. For this, we construct a class of graphs attaining the lower bound.

Before proceeding, we want to give some notions. For any two integers a and $k \geq 1$, denote by $[a]_k$ an integer such that $1 \leq [a]_k \leq k$ and $a \equiv [a]_k$ $(mod\ k)$. For a cycle $C = x_1x_2x_3 \ldots x_{k-1}x_kx_1$, we denote three special segments of C by $x_aCx_b = x_ax_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}x_b$, $\hat{x}_aCx_b = x_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}x_b$ and $\hat{x}_aC\hat{x}_b = x_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}$, where $1 \leq a, b \leq k$. Denote by $|C|$ and $|P|$ the lengths of a cycle C and a path P, respectively.

Lemma 2.3. *For a positive integer $k \neq 2$, let $C = x_1y_1x_2y_2 \ldots x_{2k}y_{2k}x_1$ be a cycle of length $4k$. Add k new vertices $z_1, z_2, \ldots, z_k$ to C, and join z_i to x_i and x_{i+k}, for $1 \leq i \leq k$. The resulting graph is denoted by H. Then, the 3-connectivity of H is 2, namely, $\kappa_3(H) = 2$.*

Proof. Since $\delta(H) = 2$, by Lemma 2.1 we can get $\kappa_3(H) \leq 2$. So the task is to show $\kappa_3(H) \geq 2$. By the definition of the generalized connectivity, it suffices to prove that $\kappa(S) \geq 2$, for every 3-subset S of $V(H)$.

Firstly, partition $V(H)$ into three types: $V_1 = \{x_1, x_2, \ldots, x_{2k}\}$, $V_2 = \{z_1, z_2, \ldots, z_k\}$ and $V_3 = \{y_1, y_2, \ldots, y_{2k}\}$. We proceed by considering all cases of S.

Case 1: $S = \{x_a, x_b, x_c\}$, where $1 \leq a < b < c \leq 2k$.

The three vertices divide the cycle C into three segments, at least one of which has length at most $|C|/3$. Without loss of generality, we may assume that $|x_aCx_b| \leq |C|/3$, namely, $|x_bCx_a| \geq 2|C|/3$. Let $b' = [b+k]_{2k}$. Note that $|x_bCx_{b'}| = |C|/2$, and so $x_{b'} \in V(\hat{x}_bC\hat{x}_a)$.

Subcase 1.1: $x_{b'} \in V(x_cC\hat{x}_a)$. In this case, $T_1 = x_aCx_bCx_c$ and $T_2 = x_cCx_{b'}Cx_a \cup x_{b'}z_{[b]_k}x_b$ are two internally disjoint trees connecting S.

Subcase 1.2: $x_{b'} \in V(\hat{x}_bC\hat{x}_c)$. Let $a' = [a+k]_{2k}$. We can get $x_{a'} \in V(\hat{x}_bC\hat{x}_{b'})$, since $1 \leq |x_aCx_b| \leq |C|/3$, $|x_aCx_{a'}| = |C|/2$ and $|x_bCx_{b'}| = |C|/2$. Therefore, $x_{a'} \in V(\hat{x}_bC\hat{x}_c)$, and then $T_1 = x_cCx_aCx_b$ and $T_2 = x_bCx_{a'}Cx_c \cup x_{a'}z_{[a]_k}x_a$ are two internally disjoint trees connecting S.

Case 2: $S = \{z_a, z_b, z_c\}$, where $1 \leq a < b < c \leq k$.

Since $1 \leq a < b < c \leq k < a+k < b+k < c+k \leq 2k$, $x_aCx_bCx_c$ and $x_{a+k}Cx_{b+k}Cx_{c+k}$ are two disjoint segments of C. It is easy to find two internally disjoint trees connecting S: $T_1 = z_ax_aCx_bCx_cz_c \cup x_bz_b$ and $T_2 = z_ax_{a+k}Cx_{b+k}Cx_{c+k}z_c \cup x_{b+k}z_b$.

Case 3: $S = \{x_a, x_b, z_c\}$, where $1 \leq a < b \leq 2k$ and $1 \leq c \leq k$.

Observe that the two neighbors x_c and x_{c+k} of z_k divide the cycle into two segments x_cCx_{c+k} and $x_{c+k}Cx_c$.

Subcase 3.1: x_a and x_b lie in distinct segments. Without loss of generality, we may assume that $x_a \in V(x_cCx_{c+k})$ and $x_b \in V(x_{c+k}Cx_c)$. Now $T_1 = x_aCx_{c+k}Cx_b \cup x_{c+k}z_c$ and $T_2 = x_bCx_cCx_a \cup x_cz_c$ are two trees we want. Note that the subcase contains the situation that either x_c or x_{c+k} is exactly x_a or x_b.

Subcase 3.2: x_a and x_b lie in the same segment. Without loss of generality, suppose that $x_a, x_b \in V(\hat{x}_cC\hat{x}_{c+k})$. Let $b' = [b+k]_{2k}$. Since $|x_cCx_{c+k}| = |C|/2$, $|x_bCx_{b'}| = |C|/2$ and $x_b \in V(\hat{x}_cC\hat{x}_{c+k})$, we have $x_{b'} \in V(\hat{x}_{c+k}C\hat{x}_c)$ and $T_1 = x_aCx_bCx_{c+k}z_c$ and $T_2 = x_bz_{[b]_k}x_{b'}Cx_cCx_a \cup x_cz_c$ are two internally disjoint trees connecting S.

Case 4: $S = \{x_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Since $1 \leq b < c \leq k < b+k < c+k \leq 2k$, the two neighbors x_b, x_{b+k} of z_b, together with two neighbors x_c, x_{c+k} of z_c divide the cycle into four segments x_bCx_c, x_cCx_{b+k}, $x_{b+k}Cx_{c+k}$ and $x_{c+k}Cx_b$. Actually, it is easy to see that no matter which segment x_a lies in, the situations are equivalent. Therefore, without loss of generality, we may assume that $x_a \in V(x_bCx_c)$. We have $T_1 = x_aCx_cCx_{b+k}z_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cx_bCx_a \cup x_bz_b$ are two internally disjoint trees connecting S. Note that this case includes the situation that x_a is exactly x_b or x_c.

Next we consider the cases in which S contains the vertices in V_3.

Case 5: $S = \{y_a, y_b, y_c\}$, where $1 \leq a < b < c \leq 2k$.

Clearly, in this case, k is a positive integer at least 3. Among the three segments y_aCy_b, y_bCy_c and y_cCy_a of C, at least one of them has length not more than $|C|/3$. We may assume that $|y_aCy_b| \leq |C|/3 = 4k/3$. Moreover, observe that x_{a+1} lies between y_a and y_b. We have $y_b \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$, since $|x_{a+1}Cy_b| < |y_aCy_b| \leq 4k/3$ and $|x_{a+1}Cx_{[a+1+k]_{2k}}| = |C|/2 = 2k$.

Subcase 5.1: $y_c \in V(\hat{y}_bC\hat{x}_{[a+1+k]_{2k}})$. There is at least one vertex x_{b+1} between y_b and y_c. Since $x_{b+1} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$, it is clear that $x_{[b+1+k]_{2k}} \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{x}_{a+1})$, namely, $x_{[b+1+k]_{2k}} \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{y}_a)$. We can find two internally disjoint trees connecting S: $T_1 = y_ax_{a+1}Cy_b \cup y_cCx_{[a+1+k]_{2k}} \cup x_{a+1}z_{[a+1]_k}x_{[a+1+k]_{2k}}$ and $T_2 = y_bx_{b+1}Cy_c \cup x_{b+1}z_{[b+1]_k}x_{[b+1+k]_{2k}}Cy_a$.

Subcase 5.2: $y_c \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{y}_a)$. There is at least one vertex x_a between y_c and y_a. Obviously, $x_{[a+k]_{2k}} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$. Moreover, $x_aCy_b = |y_aCy_b| + 1 \leq |C|/3 + 1 = 4k/3 + 1$ and $x_aCx_{[a+k]_{2k}} = |C|/2 = 2k$, where $k \geq 3$. So $y_b \in V(\hat{x}_aC\hat{x}_{[a+k]_{2k}})$. Now $T_1 = y_ax_{a+1}Cy_b \cup x_{a+1}z_{[a+1]_k}x_{[a+1+k]_{2k}}Cy_c$ and $T_2 = y_bCx_{[a+k]_{2k}}z_{[a]_k}x_a \cup y_cCx_ay_a$ are two internally disjoint trees connecting S.

Case 6: $S = \{y_a, y_b, x_c\}$, where $1 \leq a < b \leq 2k$ and $1 \leq c \leq 2k$.

Notice that y_a and y_b divide C into two segments y_aCy_b and y_bCy_a. Let $c' = [c+k]_{2k}$, and then two subcases arise.

Subcase 6.1: x_c and $x_{c'}$ lie in distinct segments. We may assume that $x_c \in V(y_aCy_b)$ and $x_{c'} \in V(y_bCy_a)$. Thus, $T_1 = y_aCx_cCy_b$ and $T_2 = y_bCx_{c'}Cy_a \cup x_cz_{[c]_k}x_{c'}$ are exactly two trees we want.

Subcase 6.2: x_c and $x_{c'}$ lie in the same segment. Without loss of generality, we may assume that $x_c, x_{c'} \in V(y_bCy_a)$ and they occur in cyclic order $y_a, y_b, x_c, x_{c'}$ on C. The segment y_aCy_b must contain a vertex x_{a+1} in V_1. Since $x_{a+1} \in V(\hat{x}_{c'}C\hat{x}_c)$, $x_{[a+1+k]_{2k}} \in$

$V(\hat{x}_c C \hat{x}_{c'})$. So we can find two internally disjoint trees connecting S: $T_1 = y_a x_{a+1} C y_b \cup x_{a+1} z_{[a+1]_k} x_{[a+1+k]_{2k}} \cup x_c C x_{[a+1+k]_{2k}}$ and $T_2 = y_b C x_c z_{[c]_k} x_{c'} C y_a$.

Case 7: $S = \{y_a, y_b, z_c\}$, where $1 \le a < b \le 2k$ and $1 \le c \le k$.

If $k = 1$, then $C = x_1 y_1 x_2 y_2 x_1$ and $H = C \cup x_1 z_1 x_2$. So y_a, y_b and z_c are exactly y_1, y_2 and z_1, respectively. Now $T_1 = y_2 x_1 y_1 \cup x_1 z_1$ and $T_2 = y_1 x_2 y_2 \cup x_2 z_1$ are two internally disjoint trees connecting S.

Otherwise, $k \ge 3$, since $k \ne 2$. We know that y_a, y_b divide C into two segments $y_a C y_b, y_b C y_a$, and z_c has two neighbors x_c and x_{c+k}.

Subcase 7.1: x_c and x_{c+k} lie in distinct segments. Suppose that $x_c \in V(y_a C y_b)$ and $x_{c+k} \in V(y_b C y_a)$. Clearly $T_1 = y_a C x_c C y_b \cup x_c z_c$ and $T_2 = y_b C x_{c+k} C y_a \cup x_{c+k} z_c$ are two internally disjoint trees connecting S.

Subcase 7.2: x_c and x_{c+k} lie in the same segment. Without loss of generality, we may assume that $x_c, x_{c+k} \in V(y_b C y_a)$ and they occur in cyclic order y_a, y_b, x_c, x_{c+k} on C.

Subsubcase 7.2.1: Between y_a and y_b, there are at least two vertices in V_1. Clearly $x_{a+1} \ne x_b$, and $y_a, x_{a+1}, x_b, y_b, x_c, x_{[a+1+k]_{2k}}, x_{[b+k]_{2k}}$ and x_{c+k} are the cyclic order in which they occur on C. So we can find two internally disjoint trees connecting S: $T_1 = y_a x_{a+1} z_{[a+1]_k} x_{[a+1+k]_{2k}} \cup y_b C x_c C x_{[a+1+k]_{2k}} \cup x_c z_c$ and $T_2 = y_b x_b z_{[b]_k} x_{[b+k]_{2k}} C x_{c+k} C y_a \cup x_{c+k} z_c$.

Subsubcase 7.2.2: Between y_a and y_b, there is only one vertex in V_1, i.e, $x_{a+1} = x_b$. Let $b' = [b+k]_{2k}$ and clearly $x_{b'} \in V(\hat{x}_c C \hat{x}_{c+k})$. Since $k \ge 3$, $V(\hat{x}_c C \hat{x}_{c+k})$ contains at least two vertices x_{c+1}, x_{c+k-1} in V_1. If $x_{c+1} \ne x_{b'}$, then $x_{[c+1+k]_{2k}} = x_{[c+k+1]_{2k}} \ne x_b \in V(\hat{x}_{c+k}) C \hat{y}_a$. So $T_1 = y_a x_b y_b \cup x_b z_{[b]_k} x_{b'} C x_{c+k} z_c$ and $T_2 = y_b C x_c y_c x_{c+1} z_{[c+1]_k} x_{[c+k+1]_{2k}} C y_a \cup x_c z_c$ are two internally disjoint trees connecting S. Otherwise, $x_{c+k-1} \ne x_{b'}$, i.e, $x_{[c-1]_{2k}} \ne x_b$. We have $x_{[c-1]_{2k}} \in V(\hat{y}_b C \hat{x}_c)$. So $T_1 = y_a x_b y_b \cup x_b z_{[b]_k} x_{b'} \cup z_c x_c C x_{b'}$ and $T_2 = y_b C x_{[c-1]_{2k}} z_{[c-1]_k} x_{c+k-1} y_{c+k-1} x_{c+k} C y_a \cup x_{c+k} z_c$ are two internally disjoint trees connecting S.

Case 8: $S = \{y_a, x_b, x_c\}$, where $1 \le a \le 2k$ and $1 \le b < c \le 2k$.

Let $b' = [b+k]_{2k}$ and $c' = [c+k]_{2k}$. If $b' = c$, i.e., $c = [b+k]_{2k}$, then without loss of generality, we may assume that $y_a \in V(x_b C x_c)$. We have $T_1 = y_a C x_c z_{[c]_k} x_b$ and $T_2 = x_c C x_b C y_a$ are two internally disjoint trees connecting S. Otherwise, $b' \ne c$. Without loss of generality, suppose $x_b, x_c, x_{b'}$ and $x_{c'}$ are the cyclic order in which they occur on C, and then they divide C into four segments $x_b C x_c, x_c C x_{b'}, x_{b'} C x_{c'}$ and $x_{c'} C x_b$.

Subcase 8.1: $y_a \in V(x_b C x_c)$. We can find two internally disjoint trees connecting S: $T_1 = x_b C y_a \cup x_c C x_{b'} z_{[b]_k} x_b$ and $T_2 = y_a C x_c z_{[c]_k} x_{c'} C x_b$.

Subcase 8.2: $y_a \in V(x_cCx_{b'})$ or $y_a \in V(x_{c'}Cx_b)$. It is easy to see that the two situations are actually equivalent. So we only consider the former. We can find two internally disjoint trees connecting S: $T_1 = x_bCx_cCy_a$ and $T_2 = y_aCx_{b'}Cx_{c'}z_{[c]_k}x_c \cup x_{b'}z_{[b]_k}x_b$.

Subcase 8.3: $y_a \in V(x_{b'}Cx_{c'})$. We can find two internally disjoint trees connecting S: $T_1 = x_bCx_c \cup x_bz_{[b]_k}x_{b'}Cy_a$ and $T_2 = y_aCx_{c'}Cx_b \cup x_{c'}z_{[c]_k}x_c$.

Case 9: $S = \{y_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Observe that x_b, x_c, x_{b+k} and x_{c+k} divide the cycle into four segments x_bCx_c, x_cCx_{b+k}, $x_{b+k}Cx_{c+k}$ and $x_{c+k}Cx_b$. Actually, no matter which segment y_a lies in, the situations are equivalent. So without loss of generality, we may assume that $y_a \in V(x_bCx_c)$. Now $T_1 = y_aCx_cCx_{b+k}z_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cx_bCy_a \cup x_bz_b$ are two internally disjoint trees connecting S.

Case 10: $S = \{y_a, x_b, z_c\}$, where $1 \leq a \leq 2k$, $1 \leq b \leq 2k$ and $1 \leq c \leq k$.

Subcase 10.1: $b = c$ or $b = c + k$. Without loss of generality, we may assume that $b = c$ and $y_a \in V(x_{c+k}Cx_b)$. Therefore, $T_1 = y_aCx_bz_c$ and $T_2 = x_bCx_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

Subcase 10.2: $b \neq c$ and $b \neq c+k$. Let $b' = [b+k]_{2k}$. We may assume that $x_b, x_c, x_{b'}$ and x_{c+k} are the cyclic order in which they occur on C. Moreover, they divide C into four segments x_bCx_c, $x_cCx_{b'}$, $x_{b'}Cx_{c+k}$ and $x_{c+k}Cx_b$.

If $y_a \in V(x_bCx_c)$, then $T_1 = y_aCx_cCx_{b'}z_{[b]_k}x_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cx_bCy_a$ are two internally disjoint trees connecting S.

If $y_a \in V(x_cCx_{b'}Cx_{c+k})$, then $T_1 = x_bCx_cCy_a \cup x_cz_c$ and $T_2 = y_aCx_{c+k}Cx_b \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

If $y_a \in V(x_{c+k}Cx_b)$, then $T_1 = y_aCx_bCx_cz_c$ and $T_2 = x_bz_{[b]_k}x_{b'}Cx_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

The proof is complete. ∎

Remark 2.3: Clearly the order $v(H)$ of the graph H is $5k$ and the size $e(H)$ is $4k+2k = 6k$, where $k \neq 2$ is a positive integer. Therefore $e(H) = \frac{6}{5}v(H)$, and by Lemma 2.3, we know that $\kappa_3(H) = 2$. It follows that H attains the lower bound of Proposition 2.1.

Remark 2.4: If $k = 2$, then H is a connected graph of order 10 and size 12. By Lemma 2.2, we can get $\kappa_3(H) = 1$. This is the reason why we add the condition $k \neq 2$ to Lemma 2.3. Moreover, no graphs of order 10 can attain the lower bound.

Now, we can obtain our main result.

Theorem 2.2. *If G is a graph of order n with $\kappa_3(G) = 2$, then $e(G) \geq \frac{6}{5}n$ and the lower bound is sharp.* ∎

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